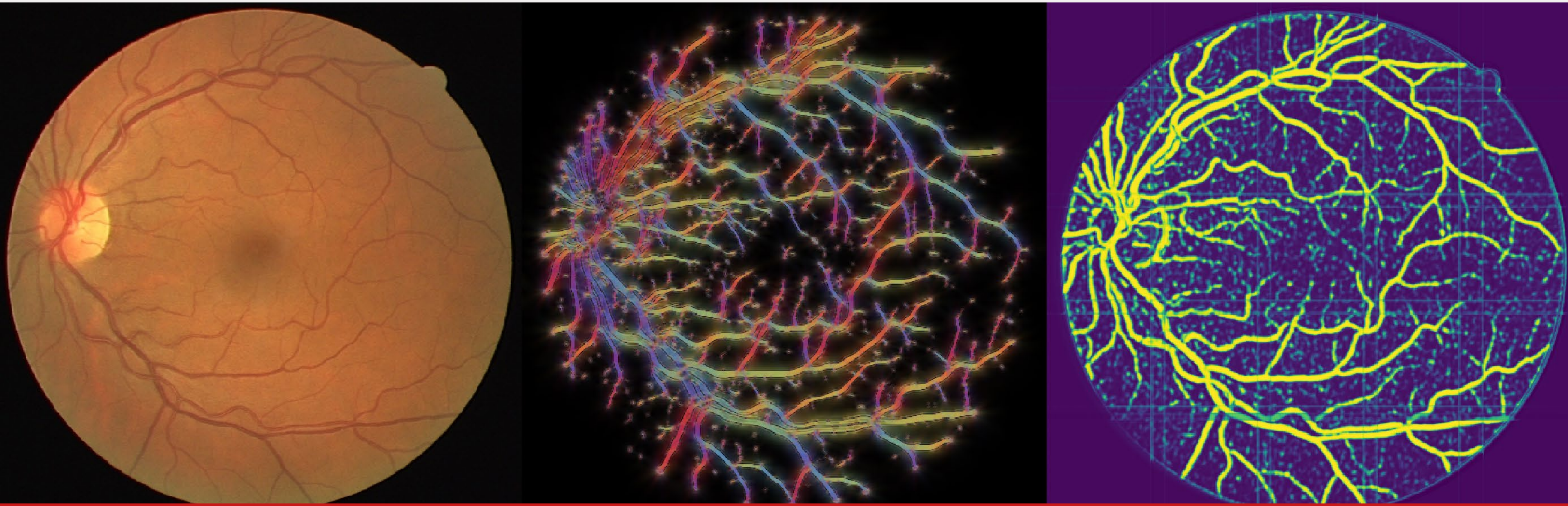

PDE-based CNNs

with Morphological Convolutions

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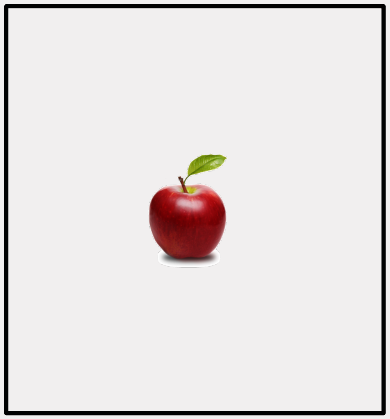
Basic Example

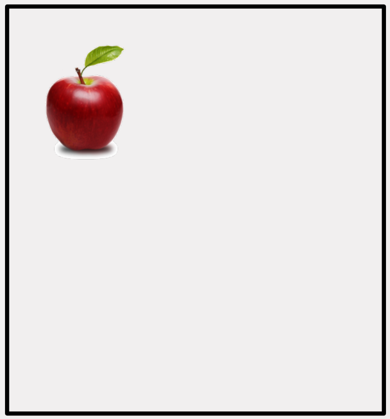
Modeling a heat process by a single layer CNN

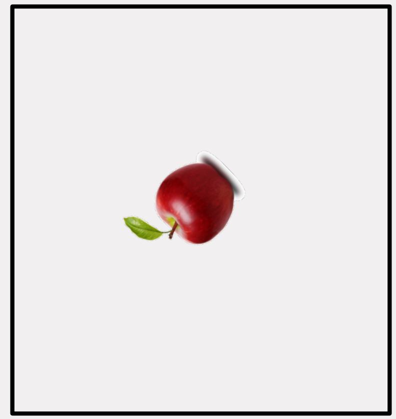
Heat equation	$\equiv$	**Analytic solution**	$\equiv$	**Single layer CNN**
$\begin{cases} \frac{\partial f}{\partial t} = \nabla^2 f \\ f(0) = f_0 \end{cases}$		$f(t) = G_t * f_0$ with G_t the heat kernel for time t		$f_{out} = \sigma(K * f_0)$ With σ a ReLU and K a learnable kernel

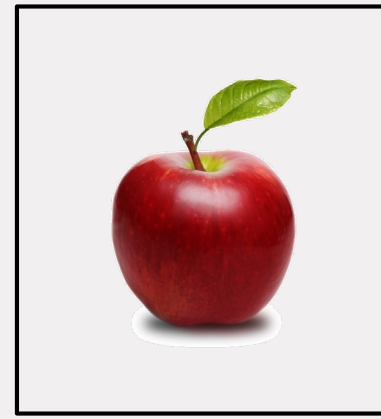
Equivariance

Group CNNs


“Apple”


“Apple”

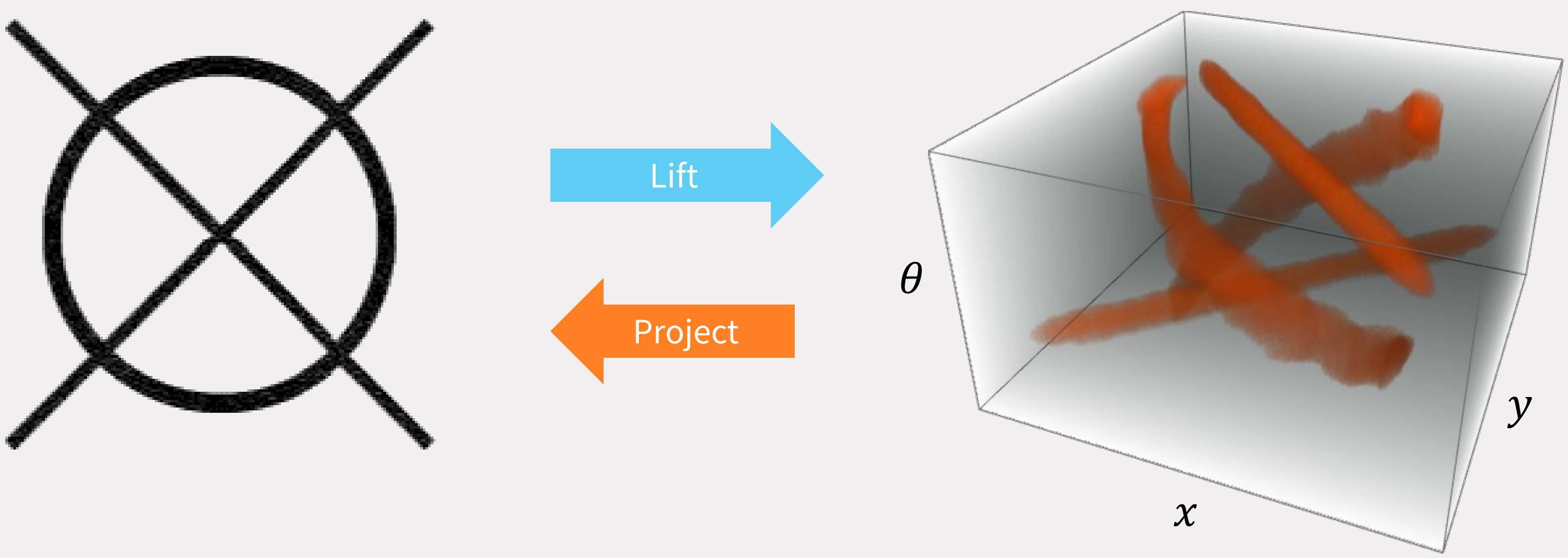

“Dog”


“Boat”

	Translation	Rotation	Scaling
Spatial CNN	✓		
SE(d) CNN	✓	✓	
SIM(d) CNN	✓	✓	✓

Extending the domain

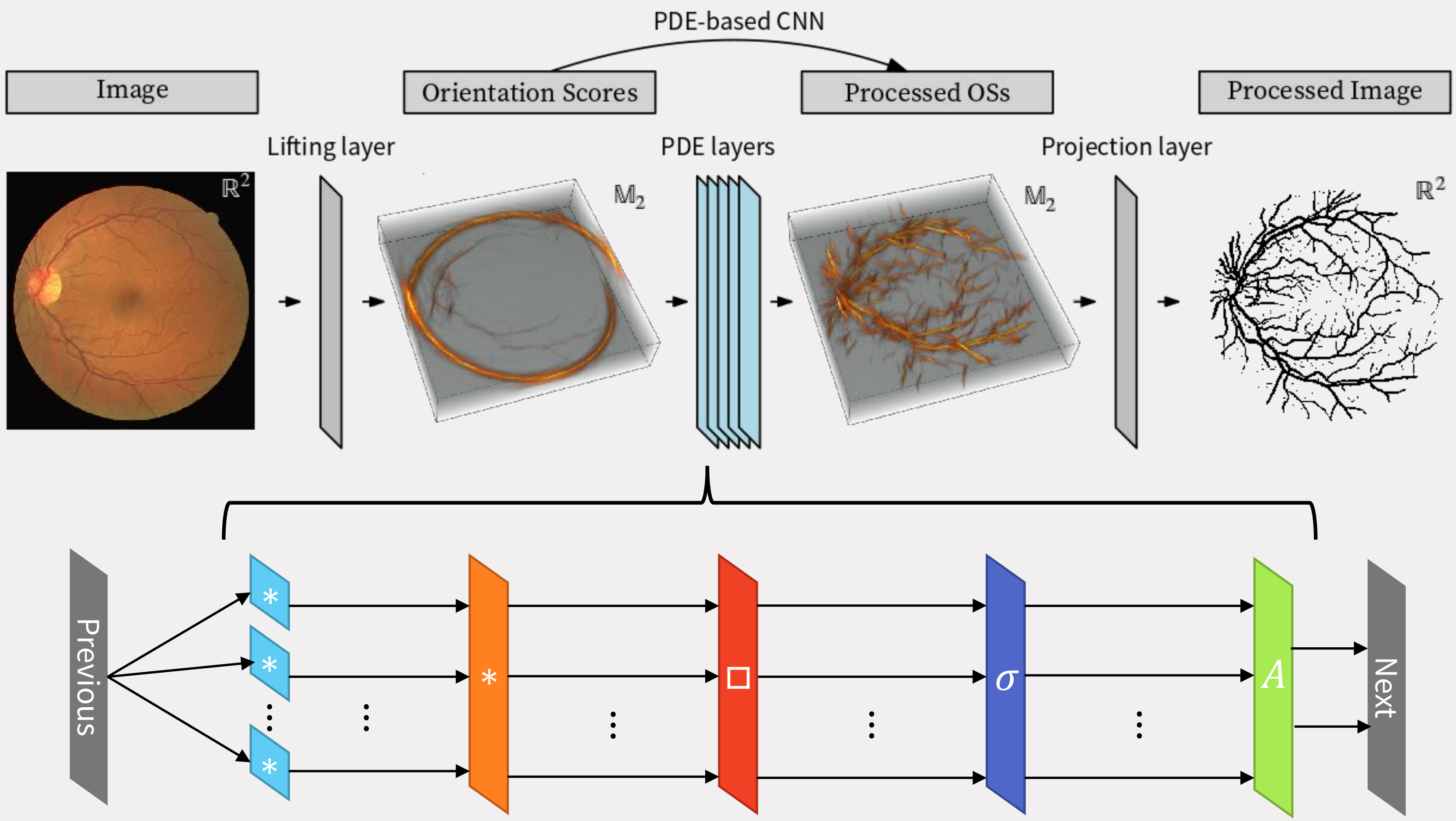
To a homogeneous space of the symmetry group



- Orientation Score Transform
- Can be learned
- More straightforward to design roto-translation equivariant CNNs

PDE-based CNN

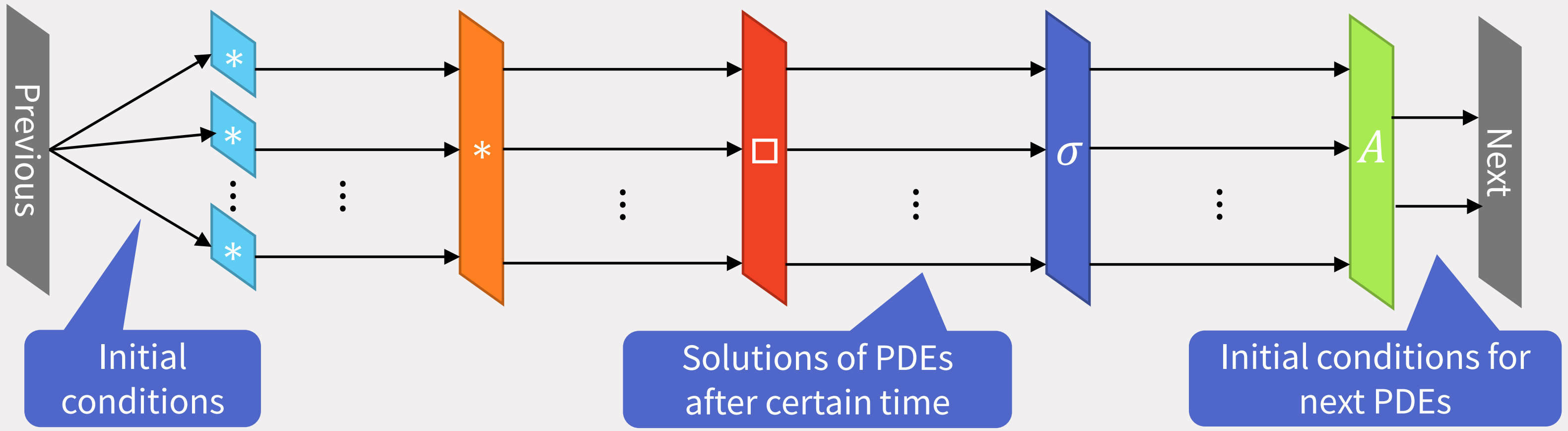
An example segmentation network



PDE Layer

$$\left(\frac{\partial W_k}{\partial t} = -\boldsymbol{c}_k \cdot \nabla W_k - |\Delta|^{\alpha} W_k \pm \|\nabla W_k\|^{2\alpha} \quad , W_k(\cdot, 0) = U_k(\cdot)\right)_{k=1}^{K}$$

Goal	Transport	Regularization	Max pooling	Normalization	Combination
PDE	Convection	Fractional diffusion	Dilation Erosion	Codomain transformation	Create initial conditions for next set of PDEs
Numerical operation	Convolution		Morph. Convolution	ReLU	Linear combinations



Parameters

What will we be training?

$$\frac{\partial W_k}{\partial t} = -\boldsymbol{c}_k \cdot \nabla_{\mathrm{G}_1} W_k - \left|\Delta_{G_2}\right|^{\alpha} W_k \pm \left\|\nabla_{\mathrm{G}_3} W_k\right\|_{G_3}^{2\alpha}$$

- K convection vectors
- 3 metric tensor fields
  - Inducing metrics d_{G_i} on the homogeneous space

- Design parameter: $\alpha \in \left(\frac{1}{2}, 1\right]$

Constructing Solutions

Let G be a Lie group (e.g. $SE(d)$)
Let G/H be a homogeneous space (e.g. $SE(d)/(\{\mathbf{0}\} \times SO(d-1))$)
G acts on G/H by $\odot$.
Let $f \in L^2, K_1 \in L^1(G/H, \mathbb{R})$.

- Linear Convolution

$$(K_1 * f)(x) = \int_G K_1\left(h^{-1} \odot x\right) f(h \odot e)\, d\mu(h)$$

- Morphological Convolution

$$(K_2 \square f)(x) = \sup_{h \in G}/\inf \; K_2\left(h^{-1} \odot x\right) + f(h \odot e)$$

  - sup : dilation
  - inf : erosion

Max Pooling

Morphological Convolution generalizes Max Pooling

For $\alpha \downarrow {}^1/_2$ the solution kernel to the morphological part converges to

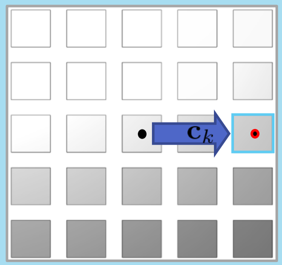
$$K_t(x) = \begin{cases} 0 & \text{if} \;\; d_{G_3}(e,x) \le t, \\ -\infty & \text{else.} \end{cases}$$

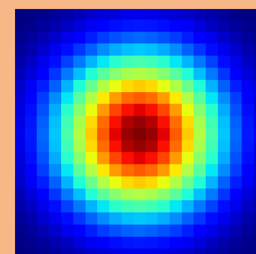
$$\begin{aligned}
(K_t \square f)(x) &= \sup_{h \in G} K\left(h^{-1} \odot x\right) + f(h \odot e) \\
&= \sup\left\{ f(h \odot e) \,\middle|\, h \in G : d_{G_3}\left(e, h^{-1} \odot x\right) \le t \right\} \\
&= \sup\left\{ f(h \odot e) \,\middle|\, h \in G : d_{G_3}(h \odot e, x) \le t \right\} \\
&= \sup_{y \in B(x,t)} f(y)
\end{aligned}$$

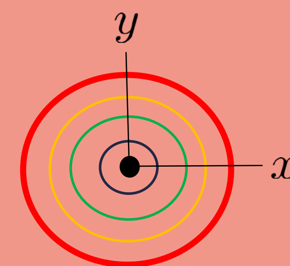
For $\alpha > {}^1/_2$: “soft” max pooling

Geometric Interpretation

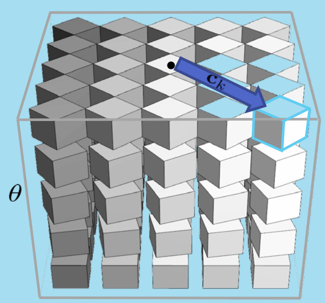
Transport	Regularization	Max Pooling

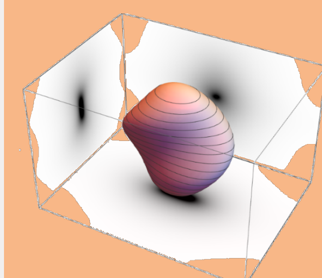


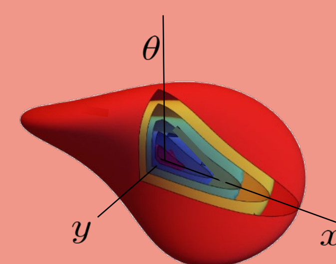




$$\frac{\partial W_k}{\partial t} = -\boldsymbol{c}_k \cdot \nabla_{G_1} W_k \; - \; \left|\Delta_{G_2}\right|^{\alpha} W_k \; \pm \; \left\|\nabla_{G_3} W_k\right\|_{G_3}^{2\alpha}$$







Maintaining Equivariance

Conditions on the kernels?

- Linear convolution

$$\forall g \in G: \quad (K * f)(g \odot x) = \Big(K * \big(y \mapsto f(g \odot y)\big)\Big)(x)$$

$$\Longleftrightarrow$$

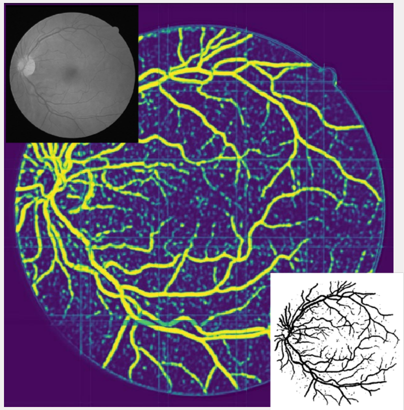
$$\forall g, h \in G \;\; \forall x \in G/H: \quad K(hg \odot x) = K(gh \odot x)$$

- Same for morphological convolution

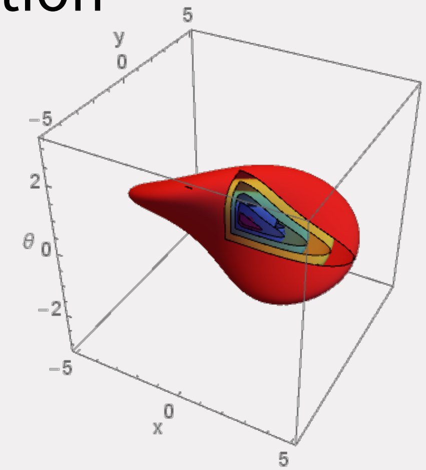
- Kernel symmetries are required!

First Experiment

Adding morphological convolution to a retinal segmentation network

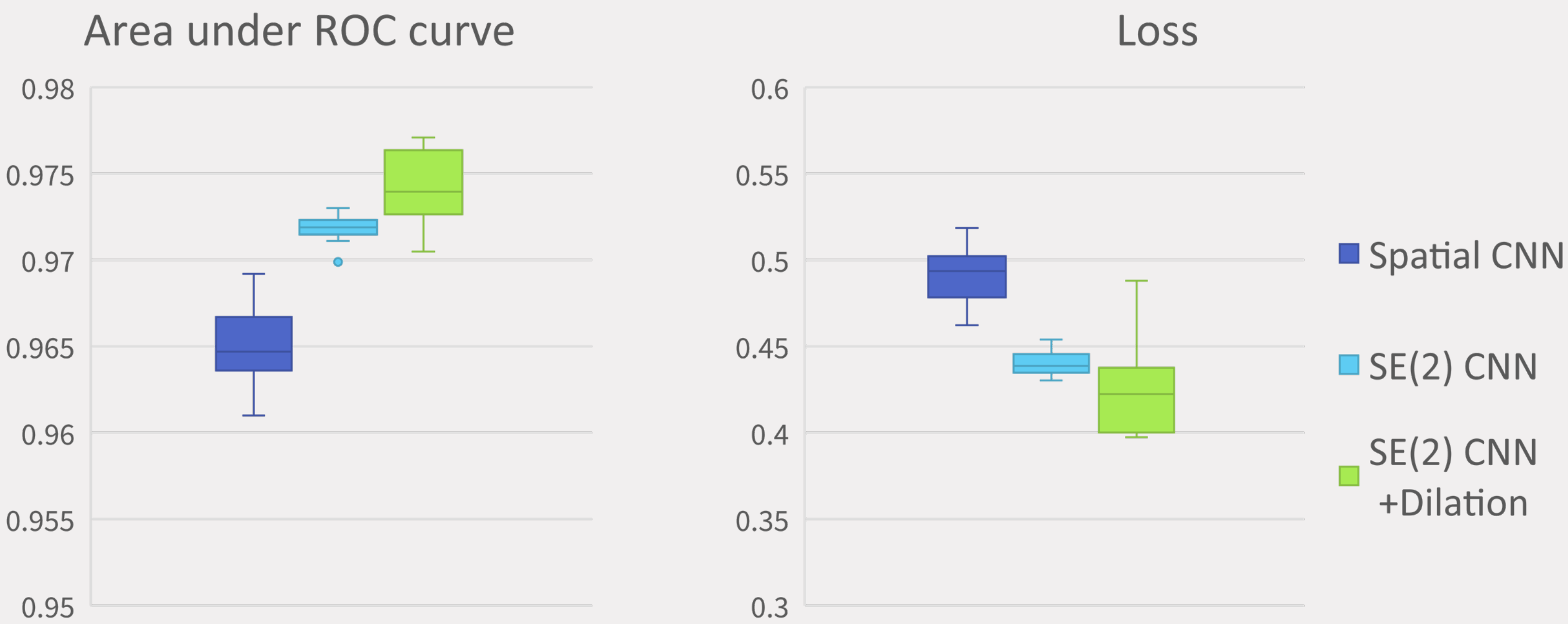


- Spatial CNN
  - 6 convolution layers
  - 34580 parameters
- SE(2) group CNN
  - Lift layer, 4 convolution layers, projection layer
  - 33916 parameters
- SE(2) group CNN with single morph. convolution
  - Lift layer, 4 convolution layers, dilation layer, projection layer
  - 33916 parameters
  - Fixed morph. convolution kernel for $\alpha = {}^{2}/_{3}$



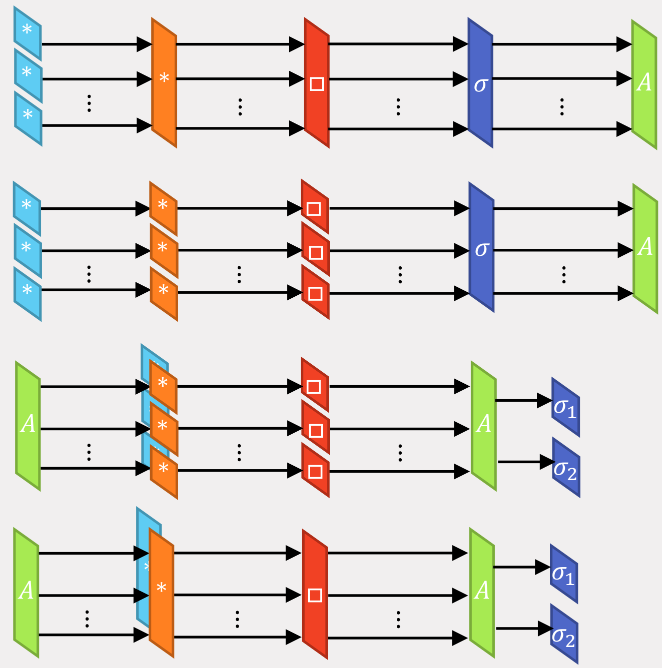
First Experiment

Performance improvement



Current and Future Work

- TensorFlow implementation
  - experiments
- Layer architectures
- Geometric interpretability
- Probabilistic interpretability
- Integrate PDE framework for geometric equivariant processing of orientation scores (2005-now)

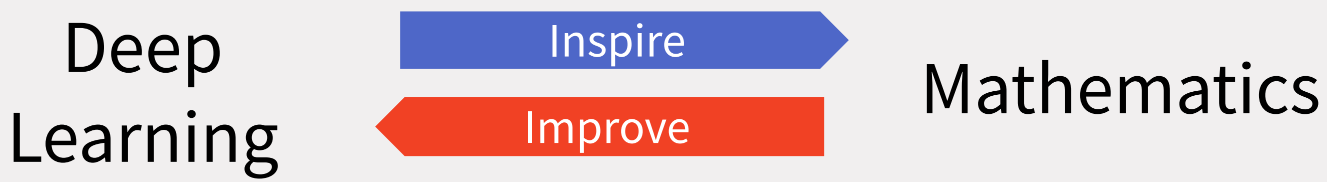


∇ Δ div $(\cdot,\cdot)$ $\|\cdot\|$

Concluding remarks

Geometric PDE framework for CNNs

- Improved performance over state-of-the-art G-CNNs
    - by inclusion of single PDE-based morphological convolution layer
- Problem symmetry integral part of the design



Also see 2 talks in IHP Paris YouTube search: “Remco Duits IHP”